Symmetrisable Functions and their Expansion in Terms of Biorthogonal Functions.

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(Communicated by Prof. E. W. Hobson, Sc.D., F.R.S. Received April 12, 1920.)

The purpose of this communication is to announce certain results relative to the expansion of a symmetrisable function $\kappa(s, t)$ in terms of a complete biorthogonal system of fundamental functions,* which belong to $\kappa(s, t)$ regarded as the kernel of a linear integral equation. An indication of the method by which the results have been obtained is given, but no attempt is made to supply detailed proofs.

Preliminary Explanations.

§ 1. Let $\kappa(s, t)$† be a function defined in the square $a \leq s \leq b$, $a \leq t \leq b$. If a function $\gamma(s, t)$ can be found which is of positive type in the square $a \leq s \leq b$, $a \leq t \leq b$, and such that

$$\int_a^b \gamma(s, x)\, \kappa(x, t)\, dx$$

is a symmetric function of s and t, $\kappa(s, t)$ is said to be *symmetrisable on the left* by $\gamma(s, t)$ in the square. Similarly, if a function $\gamma'(s, t)$ of positive type can be found such that

$$\int_a^b \kappa(s, x)\, \gamma'(x, t)\, dx$$

is a symmetric function of s and t, $\kappa(s, t)$ is said to be *symmetrisable on the right* by $\gamma'(s, t)$.‡

In general theory the function by which $\kappa(s, t)$ is symmetrisable is usually assumed to be definite.§ The result is that the theory is only applicable

* For the definition of a fundamental function, reference may be made to Goursat, 'Cours d'Analyse,' vol. 3, § 568.

† For brevity, conditions as to summability, etc., are not stated in this section. There will be no difficulty in supplying these conditions.

‡ In what follows, functions symmetrisable on the left will be considered almost exclusively.

§ Lalesco, 'Introduction à la Théorie des Équations Intégrales,' p. 78 (1912); also Goursat, 'Cours d'Analyse,' vol. 3, § 596, pp. 466–8 (1914). Marty ('Comptes Rendus,' t. 150, p. 1031), at first imposed only the condition that the function should be of positive type. He found, however, that this condition was too wide as a basis for a general theory (see footnote, *op. cit*, p. 1500). For the definition of the term "definite," *vide* Goursat, *ov. cit.*, § 5901.

to particular forms of the two most important types of symmetrisable function.* In order to remove this blemish the notion of a "completely" symmetrisable function has been introduced. Further justification for this departure from the usual practice will be found in the expansion theorems stated below.

Let $\kappa(s, t)$ be symmetrisable on the left by $\gamma(s, t)$. The function $\kappa(s, t)$ will be said to be *completely symmetrisable on the left* by $\gamma(s, t)$ if either (i) no solution of a homogeneous integral equation of the type

$$\phi(s) = \lambda \int_a^b \kappa(s, t)\, \phi(t)\, dt$$

exists; or (ii) this equation can be solved for one or more values of λ, but no solution is such that $\int_a^b \gamma(s, t)\phi(t)\, dt = 0$. In the first case $\kappa(s, t)$, considered as the kernel of a linear integral equation, has no singular value: in the second case it has at least one singular value.

As an example consider

$$\kappa(s, t) = \alpha(s)\, \gamma(s, t),$$

where $\gamma(s, t)$ is of positive type in the square $a \leq s \leq b$, $a \leq t \leq b$, and $\alpha(s)$ is a function defined in the interval (a, b). It may be verified that $\kappa(s, t)$ is completely symmetrisable on the left by $\gamma(s, t)$, and is completely symmetrisable on the right by $\alpha(s)\gamma(s, t)\alpha(t)$. This function is associated with the name of Hilbert.†

A second example is furnished by

$$\kappa(s, t) = \int_a^b \alpha(s, x)\, \gamma(x, t)\, dx,$$

where $\gamma(s, t)$ is as before, and $\alpha(s, t)$ is *any* function which is symmetric in the square $a \leq s \leq b$, $a \leq t \leq b$. In this case $\kappa(s, t)$ is completely symmetrisable on the left by $\gamma(s, t)$, and is completely symmetrisable on the right by

$$\int_a^b \int_a^b \alpha(s, x)\, \gamma(x, y)\, \alpha(y, t)\, dx\, dy.$$

This type of symmetrisable function was indicated by Marty.‡

* These are described at the end of this paragraph.

† *Vide* 'Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen, pp. 195–204 (1912). The generalisation $\alpha(s)\, \gamma(s, t)\, \beta(t)$ was pointed out by Goursat.

‡ Comptes Rendus,' t. 150, p. 1500 (1910), footnote.

§ 2. The following properties of a kernel $\kappa(s,t)$ which is completely symmetrisable on the left by $\gamma(s,t)$ can be established*:—

(i) *Either* $\int_a^b \gamma(s,x)\,\kappa(x,t)\,dx = 0$ *or* $\kappa(s,t)$ *has at least one singular value.*

(ii) *A singular value of* $\kappa(s,t)$ *is necessarily real.*

(iii) *A pole of the resolvent*† *is necessarily simple.*

(iv) *Let* $\int_a^b \gamma(sx)\,\kappa(x,t)\,dx \neq 0$, *and let the numbers* $\lambda_1, \lambda_2, \ldots \lambda_n \ldots$ *consist of the singular values of* $\kappa(s,t)$ *each repeated that number of times which corresponds to the order of the singular value considered as a zero of the determinant of* $\kappa(s,t)$. *Then, corresponding to a number* λ_n, *there is a pair of functions* $\phi_n(s)$, $\psi_n(s)$ *which satisfy the homogeneous equations*

$$\phi_n(s) = \lambda_n \int_a^b \kappa(s,t)\,\phi_n(t)\,dt,$$

$$\psi_n(t) = \lambda_n \int_a^b \psi_n(s)\,\kappa(s,t)\,ds,$$

and are such that no linear equation connects functions $\phi_n(s)$ (*or* $\psi_n(s)$) *corresponding to the same singular value. Moreover, the pairs of functions* $\phi_n(s)$, $\psi_n(s)$ *can be chosen in such a way that* (I)

$$\left.\begin{array}{l}\phi_1(s),\ \phi_2(s), \ldots, \phi_n(s), \ \ldots \\ \psi_1(s),\ \psi_2(s), \ldots, \psi_n(s), \ \ldots\end{array}\right\} \qquad (1)$$

forms a biorthogonal system of functions for the interval (a,b)‡ *and* (II) $\psi_n(s) = \mu_n \int_a^b \gamma(s,t)\,\phi_n(t)\,dt$, *where* μ_n *is a positive constant.*

The functions $\phi_n(s)$ can be so chosen that μ_n is unity, but it appears to be advisable not to do this, in order to preserve symmetry in statements of certain results. In the case of a function completely symmetrisable on both sides there will be a number μ_n', where $\phi_n(s) = \mu_n' \int_a^b \gamma'(s,t)\,\psi_n(t)\,dt$, and it is supposed that the function is symmetrisable on the right by $\gamma'(s,t)$. It is possible to choose the biorthogonal system in such a way that μ_n is always unity, or that μ_n' is always unity. In general, however, it is not possible to secure that μ_n and μ_n' are *both* unity.§

* The proofs follow closely those given by Lalesco, *op. cit.*, pp. 80–82.

† Reciprocal function, solving function.

‡ It is understood by this that $\int_b^a \phi_n(s)\,\psi_m(s)\,ds = 0$ or 1, according as $n \neq m$ or $n = m$.

§ Lalesco (*op. cit.*, p. 84) appears to have overlooked this fact.

The functions (1) having the properties stated in (iv) will be called a *complete biorthogonal system of fundamental functions* for the kernel $\kappa(s, t)$.

§ 3. The results stated in the preceding paragraph may be regarded as generalisations of the corresponding well-known results for a symmetric function.* It is therefore natural to enquire how far the expansion theorems (*e.g.*, those of Hilbert and Schmidt) connected with the symmetric function can be extended to the more general class. The most important published results in this direction have been obtained by Pell† and Garbe.‡ The first writer develops a theory of biorthogonal systems of functions, and upon this basis establishes an expansion theorem. From the point of view here adopted the most interesting feature of the theorem is its application to the function $\kappa(s, t) = \int_a^b \alpha(s, x)\gamma(x, t)\,dx$. It follows, in fact, that when $\kappa(s, t)$ is of this form any function which can be expressed as $\int_a^b \kappa(s, t) g(t)\,dt$ admits of an expansion

$$f(s) = \left\{\phi_1(s)\int_a^b f(t)\psi_1(t)\,dt + \phi_2(s)\int_a^b f(t)\psi_2(t)\,dt + \right.$$
$$\left. \ldots + \phi_n(s)\int_a^b f(t)\psi_n(t)\,dt + \ldots\right\} + h(s),$$

where $h(s)$ is such that $\int_a^b \gamma(s, t)h(t)\,dt = 0$, and the series in the brackets is absolutely and uniformly convergent. The function $h(s)$ may be zero, and certainly will be so when $\gamma(s, t)$ is definite.

Garbe§ proves that when $\kappa(s, t) = \alpha(s)\gamma(s, t)$ the statements just made are also valid. He then restricts $\gamma(s, t)$ to be "allgemein"|| and proves that under this condition $\kappa(s, t)$ can be expanded in the form¶

$$\frac{\phi_1(s)\psi_1(t)}{\lambda_1} + \frac{\phi_2(s)\psi_2(t)}{\lambda_2} + \ldots + \frac{\phi_n(s)\psi_n(t)}{\lambda_n} + \ldots.$$

* A symmetric function $\kappa(s, t)$ is completely symmetrisable on either side by the iterated function $\kappa_2(s, t)$.

† "Applications of Biorthogonal Systems of Functions to the Theory of Integral Equations," 'Trans. Amer. Math. Soc.,' 1911, p. 173, §4. The condition (c_1), p. 167, and the footnote on p. 165, should be noted in connection with the definition of a completely symmetrisable function given above. In virtue of this condition the theorem lacks generality in its statement.

‡ 'Mathematische Annalen,' vol. 76, pp. 533–4 (1914–15).

§ Garbe does not impose any restriction upon $\gamma(s, t)$.

|| Hilbert, *op. cit.*, p. 25. The term is somewhat misleading, since "allgemein" kernels are a very restricted class.

¶ Pp. 538–542, Garbe states the result in a slightly different form.

A result of this kind is suggested by the known expansion of a symmetric function of positive type in terms of its fundamental functions.* It is easy to see, however, that an expansion which contains *only* terms of the form $\phi_n(s)\psi_n(t)/\lambda_n$ is not valid for all symmetrisable functions, not even for the comparatively simple ones of the Hilbert type. In the following paragraph an example is given in justification of this statement. The example will also serve as an existence theorem in connection with the general forms of expansion given below.

§ 4. *Example.*

Let $a = 0$, $b = \pi$; and let $\phi_r(s)$, $\psi_r(s)$, $\xi_r(s)$, $\eta_r(s)$, $(r = 1, 2)$ be functions defined as follows:—

$$\psi_1(s) = \sqrt{1/\pi}\,(1+\cos 3s),$$
$$\psi_2(s) = \sqrt{2/\pi}\,(\cos 7s+\cos 10s),$$
$$\eta_1(s) = \sqrt{2/\pi}\cos s,$$
$$\eta_2(s) = \sqrt{2/\pi}\cos 5s,$$
$$\phi_n(s) = \alpha(s)\psi_n(s)\ (n = 1, 2),$$
$$\xi_n(s) = \alpha(s)\eta_n(s)\ (n = 1, 2),$$

where $\alpha(s) = \cos 3s$. Then it may be verified that

$$\left.\begin{matrix} \phi_1(s), & \phi_2(s) \\ \psi_1(s), & \psi_2(s) \end{matrix}\right\}, \tag{2}$$

is a biorthogonal system of functions for the interval $(0, \pi)$; that $\eta_1(s)\,\eta_2(s)$ is an orthogonal system of functions for the same interval;† and that each of the pairs of functions $\{\phi_n(s), \eta_m(s)\}$, $\{\psi_n(s), \xi_m(s)\}$, $\{\xi_n(s), \eta_m(s)\}$ is orthogonal for the interval, whatever be the value of n and m.‡ It will be found that $\xi_1(s)$ and $\xi_2(s)$ are not orthogonal.

Let

$$\gamma(s, t) = \sum_{n=1}^{2}\frac{\psi_n(s)\psi_n(t)}{\lambda_n} + \sum_{n=1}^{2}\frac{\eta_n(s)\eta_n(t)}{v_n}, \tag{3}$$

and let

$$\kappa(s, t) = \sum_{n=1}^{2}\frac{\phi_n(s)\psi_n(t)}{\lambda_n} + \sum_{n=1}^{2}\frac{\xi_n(s)\eta_n(t)}{v_n}, \tag{4}$$

* 'Phil. Trans,' Series A, vol. 209, pp. 439–446.

† It is understood by this that $\int_a^b \eta_n(s)\,\eta_m(s)\,ds = 0$ or 1, according as $n \neq m$ or $n = m$.

‡ *I.e.*, $\int_b^a \phi_n(s)\,\eta_m(s)\,ds = \int_a^b \psi_n(s)\,\xi_m(s)\,ds = \int_a^b \xi_n(s)\,\eta_m(s)\,ds = 0$ for $n = 1, 2$, and $m = 1, 2$.

where λ_1, λ_2, ν_1, ν_2 are any real positive members. It will be clear that $\kappa(s,t) = \alpha(s)\gamma(s,t)$, where $\gamma(s,t)$ is of positive type in the square $0 \leq s \leq \pi$, $0 \leq t \leq \pi$. Inspection of the expression which defines $\kappa(s,t)$ will show that it consists of four kernels, any one of which is orthogonal* to each of the other three. From this it follows that the resolvent corresponding to $\kappa(s,t)$ is

$$\mathrm{K}(\lambda\,;\; s,\, t) = \sum_{n=1}^{2} \frac{\phi_n(s)\,\psi_n(t)}{\lambda_n - \lambda} + \sum_{n=1}^{2} \frac{\xi_n(s)\,\eta_n(t)}{\nu_n},$$

and hence that the system (2) is a complete biorthogonal system of fundamental functions for $\kappa(s,t)$. It will now be seen that $\kappa(s,t)$ contains terms $\xi_n(s)\,\eta_n(t)/\nu_n$ which are of a character totally different from that of the terms $\phi_n(s)\,\psi_n(t)/\lambda_n$.

§ 5. A more precise account of the results which have been obtained can now be given. In § 7 below it is shown that, if $\kappa(s,t)$ is completely symmetrisable on the left by $\gamma(s,t)$, then $\gamma(s,t)$ admits of an expansion of the form (3). In general the expansion will contain an infinite number of terms of the type $\psi_n(s)\,\psi_n(t)/\mu_n$,† and an infinite number of terms of the type $\eta_n(s)\,\eta_n(t)/\nu_n$. When either or both sets of terms is infinite, it is shown that, under very general conditions, the infinite series corresponding to such a set is absolutely and uniformly convergent.

The expansion theorem for $\gamma(s,t)$ is then applied to the particular cases (i) $\kappa(s,t) = \alpha(s)\gamma(s,t)$, (ii) $\kappa(s,t) = \int_a^b \alpha(s,x)\,\gamma(x,t)\,dx$, and it is shown that in each case $\kappa(s,t)$ admits of an expansion of the form (4). As before, the number of terms of each type is, in general, infinite, and the convergence of the series of terms of either type is, under general conditions, absolutely and uniformly convergent.

The Expansion Theorems.

§ 6. Let $\kappa(s,\,t)$ be a symmetric function which is bounded and summable in the square $a \leq s \leq b$, $a \leq t \leq b$. The function will be said to be of positive type in the square if

$$\int_a^b \int_a^b \kappa(s,\,t)\,f(s)\,f(t)\,ds\,dt \geq 0$$

for each function $f(s)$, which is bounded and summable in the interval $a \leq s \leq b$. With this definition the expansion theorem for functions of positive type as given in 'Philosophical Transactions' Series A, vol. 209, pp. 415–446, may be generalised. The following theorem will be of use hereafter :—

* Lalesco (*op. cit.*, p. 40) may be consulted.

† μ_n is positive, but is not necessarily a singular value of $\kappa(s,\,t)$.

Let $\kappa(s, t)$ be a symmetric function of positive type which is bounded and summable in the square $a \leq s \leq b$, $a \leq t \leq b$; and let the function be continuous with respect to s in the interval (a, b) for any fixed value of t in the same interval. Let $\psi_1(s), \psi_2(s), \ldots, \psi_n(s), \ldots$ be a complete orthogonal system of fundamental functions for the kernel $\kappa(s, t)$ corresponding respectively to singular values $\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots$† Then the series*

$$\sum_{n=1} \frac{\psi_n(s)\psi_n(t)}{\lambda_n}, \tag{5}$$

converges absolutely at every point of the square $a \leq s \leq b$, $a \leq t \leq b$, and has $\kappa(s, t)$ for its sum function.

Further, if $\kappa(s, s)$ is a continuous function of s in the interval (a, b), the series (5) is uniformly convergent in the square $a \leq s \leq b$, $a \leq t \leq b$, and $\kappa(s, t)$, is continuous throughout the square.‡

The last part of this theorem is remarkable from the point of view of the theory of functions of real variables. It appears that, if $\kappa(s, t)$ is a symmetric bounded and summable function of positive type such that (i) $\kappa(s, t)$ is continuous with respect to s in (a, b), for any fixed value of t in the interval, and (ii) $\kappa(s, s)$ is a continuous function of s in (a, b), then $\kappa(s, t)$ is a continuous function (of *two* variables) in the square $a \leq s \leq b$, $a \leq t \leq b$.

§ 7. Let $\kappa(s, t)$ be a function (not necessarily symmetrical) defined in the square $a \leq s \leq b, a \leq t \leq b$, which is symmetrisable on the left by $\gamma(s, t)$. It will be assumed that $\kappa(s, t)$ and $\gamma(s, t)$ are each bounded and summable in the square, and that each is a continuous function of s (t)§ in the interval (a, b) for any fixed value of t (s)§ in the interval. It should be observed that under the hypotheses stated, the functions of the complete biorthogonal system (1) § 2 are all continuous in (a, b).

The first step is to obtain an expansion theorem for $\gamma(s, t)$. For this we make use of the generalised Schwarz inequality

$$\int_a^b \int_a^b \gamma(s, t) f(s) f(t)\, ds\, dt \geq \sum_{n=1}^{m} \frac{1}{\mu_n} \left[\int_a^b f(s)\psi_n(s)\, ds\right]^2,$$

* Under the hypotheses stated, the functions are continuous; *cf.* Hobson, 'Proc. Lond. Math. Soc.,' Series 2, vol. 13, p. 308.

† These are not necessarily distinct; *cf.* § 2.

‡ Since establishing this theorem, I have discovered that it may be deduced from a theorem due to Hobson, 'Proc. Lond. Math. Soc.,' Series 2, vol. 14 (Part I), pp. 24–25. The results stated above follow from the fact that the series, $\Sigma\phi_n(s)\,\phi_n(t)/\lambda_n$, is uniformly convergent with respect to s for a fixed t. This being the case, I do not indicate my method of proof here. It is probable that the reader will be able to supply this proof after perusal of § 8.

§ It may be advisable to point out that two statements are here implied—in one the unbracketed letters are to be taken together, and in the other the bracketed letters are to be taken together.

where $f(s)$ may be assumed to be any bounded function summable in (a, b). The inequality may be written in the form

$$\int_a^b \int_a^b \left\{ \gamma(s, t) - \sum_{n=1}^{m} \frac{\psi_n(s)\,\psi_n(t)}{\mu_n} \right\} f(s) f(t)\, ds\, dt \geq 0,$$

whence it may be proved that the series

$$\sum_{n=1} \frac{\psi_n(s)\,\psi_n(t)}{\mu_n}, \tag{6}$$

is absolutely convergent in the square $a \quad s \leq b, a \leq t \leq b$.

It can be proved that $\gamma_1(s, t)$, the sum function of the series (6), is bounded and summable in the fundamental square, and that it is continuous with respect to s in the interval (a, b) for any fixed value of t in the interval: hence $\gamma_2(s, t) = \gamma(s, t) - \gamma_1(s, t)$ has the same properties. In virtue of the generalised Schwarz inequality, it will be seen that $\gamma_2(s, t)$ is of positive type. Applying the theorem of § 6 we find that

$$\gamma_2(s, t) = \sum_{n=1} \frac{\eta_n(s)\,\eta_n(t)}{v_n}, \tag{7}$$

where $\eta_1(s), \eta_2(s) \ldots, \eta_n(s), \ldots$ are a complete orthogonal system of fundamental functions for $\gamma_2(s, t)$ corresponding respectively to singular values $v_1, v_2 \ldots v_2 \ldots$, and the series on the right converges absolutely in the fundamental square. It has now been shown that

$$\gamma(s, t) = \sum_{n=1} \frac{\psi_n(s)\,\psi_n(t)}{\mu_n} + \sum_{n=1} \frac{\eta_n(s)\,\eta_n(t)}{v_n}.$$

By application of Dini's theorem* it is easy to establish that where $\gamma(s, s)$ is continuous in (a, b) the two series on the right of the equation just written converge uniformly, and hence that $\gamma_1(s, t)$, $\gamma_2(s, t)$ are both continuous functions in the fundamental square.

Since $\psi_n(s) = \mu_n \int_a^b \gamma_1(s, t)\,\phi_n(t)\,dt$ it will be seen that $\int_a^b \gamma_2(s, t)\,\phi_n(t)\,dt = 0$ for all values of n, and hence that $\int_a^b \gamma_2(s, x)\,\kappa(x, t)\,dx = 0$.† From this it is evident that $\int_a^b \eta_n(s)\,\kappa(s, t)\,ds = 0$, *i.e.*, each of the functions $\eta_1(s), \eta_2(s), \ldots, \eta_n(s), \ldots$ is orthogonal to $\kappa(s, t)$ on the left.

* *Cf.* 'Phil. Trans.,' A, vol. 209, p. 440.

† Because $\int_a^b \gamma_2(s\,x)\,\mathrm{K}(\lambda\,;\; x, t)\,dx$, where $\mathrm{K}(\lambda\,;\; s, t)$ is the resolvent, has no finite singularity when considered as a function of λ (*cf.* Marty, 'Comptes Rendus,' vol. 150, pp. 1031–3).

The following theorem can now be stated :—*Let $\kappa(s, t)$ be a function which is bounded and summable in the square $a \leq s \leq b$, $a \leq t \leq b$; and let the function be continuous with respect to s (t) in the interval (a, b) for any fixed value of t (s) in this interval. Let $\kappa(s, t)$ be completely symmetrisable on the left by a function $\gamma(s, t)$ of positive type which has the properties of boundedness, summability, and continuity postulated for $\kappa(s, t)$. Finally, let*

$$\phi_1(s),\ \phi_2(s),\ \ldots,\ \phi_n(s),\ \ldots,$$

$$\psi_1(s),\ \psi_2(s),\ \ldots,\ \psi_n(s),\ \ldots,$$

be a complete biorthogonal system of fundamental functions for the kernel $\kappa(s,t)$. Then

$$\gamma(s, t) = \sum_{n=1} \frac{\psi_n(s)\,\psi_n(t)}{\mu_n} + \sum_{n=1} \frac{\eta_n(s)\,\eta_n(t)}{v_n}, \tag{8}$$

where (i) *the functions $\eta_n(s)$ $(n = 1, 2, \ldots)$ are all continuous in the interval (a, b), and form an orthogonal system of functions for this interval,* (ii) *each function $\eta_n(s)$ is orthogonal to $\kappa(s,t)$ on the left,* (iii) *the numbers μ_n $(n=1, 2, \ldots)$ are positive and are defined by the equations*

$$\mu_n \int_a^b \int_a^b \gamma(s, t)\, \phi_n(s)\, \phi_n(t)\, ds\, dt = 1,$$

and (iv) *the numbers v_n $(n = 1, 2, \ldots)$ are positive. Each of the series on the right of* (8) *is absolutely convergent in the square $a \leq s \leq b$, $a \leq t \leq b$, and each has a sum-function which has the properties of boundedness, summability, and continuity postulated for the functions $\kappa(s, t)$, $\gamma(s, t)$.*

Further, if $\gamma(s, s)$ is a continuous function of s in the interval (a, b), each of the series on the right of (8) *is uniformly convergent in the square $a \leq s \leq b$, $a \leq t \leq b$.*

The corresponding result when $\kappa(s, t)$ is completely symmetrisable on the right by $\gamma'(s, t)$ will be obvious. The expansion for $\gamma'(s, t)$ is

$$\sum_{n=1} \frac{\phi_n(s)\,\phi_n(t)}{\mu_n'} + \sum_{n=1} \frac{\xi_n'(s)\,\xi_n'(t)}{v_n'},$$

where the functions $\xi_n'(s)$ $(n = 1, 2 \ldots)$ form an orthogonal system of functions for the interval (a, b), and each function is orthogonal to $\kappa(s, t)$ on the right.

In passing it should be observed that for the function symmetrisable on the left we have

$$\int_a^b \gamma(s, x)\, \kappa(x, t) = \int_a^b \gamma_1(s, x)\, \kappa(x, t) = \sum_{n=1} \frac{\psi_n(s)\,\psi_n(t)}{\lambda_n \mu_n},$$

and that there is a corresponding result for the case of a function symmetrisable on the right.

§ 8. From considerations of lucidity and brevity the theorems of the two preceding paragraphs have not been stated in a form which is sufficiently general in certain applications. Thus Hilbert has shown the function $\alpha(s)\gamma(st)$ to be of interest, where $\alpha(s)$ is a discontinuous function defined by $\alpha(s) = +1$ in certain sub-intervals of (a, b), and $\alpha(s) = -1$ in the complementary intervals.* To meet this and other cases an easy generalisation may be made.

Let $a_1, a_2, \ldots a_{N-1}$ be numbers which satisfy the inequalities $a < a_1 < a_2 \ldots < a_{N-1} < b$, and let a, b be denoted by a_0, a_N respectively. Let $\kappa(s, t)$ be a function which is bounded and summable in the square $a \leq s \leq b, a \leq t \leq b$: let $\kappa(s, t) = \kappa^{(p, q)}(s, t)$ $(p = 1, 2, \ldots N;\ q = 1, 2, \ldots, N)$ in the *open* rectangle $a_{p-1} < s < a_p$, $a_{q-1} < t < a_q$, where $\kappa^{(p,q)}(s, t)$ is defined in the *closed* rectangle $a_{p-1} \leq s \leq a_p$, $a_{q-1} \leq t \leq a_q$, and is (i) continuous with respect to s in the closed interval (a_{p-1}, a_p) for any fixed value of t in the closed interval (a_{q-1}, a_q); (ii) continuous with respect to t in the closed interval (a_{q-1}, a_q) for any fixed value of s in the closed interval (a_{p-1}, a_p). Let $\gamma(s, t)$ be a function of positive type which has properties similar to those postulated for $\kappa(s, t)$, and let $\kappa(s, t)$ be symmetrisable on the left by $\gamma(s, t)$. It is easily proved that, in the open interval $a_{p-1} < s < a_p$, $\phi_n(s)$ is equal to a function $\phi_n^{(p)}(s)$, which is continuous in the closed interval $a_{p-1} \leq s \leq a_p$; and similarly in regard to $\psi_n(s)$, $\eta_n(s)$. Thus, corresponding to a rectangle $a_{p-1} \leq s \leq a_p$, $a_{q-1} \leq t \leq a_q$, we have continuous functions $\kappa^{(p,q)}(s, t)$, $\gamma^{(p,q)}(s, t)$, $\phi_n^{(p)}(s)$, $\phi_n^{(q)}(t)$, $\psi_n^{(p)}(s)$, $\psi_n^{(q)}(t)$, $\eta_n^{(p)}(s)$, $\eta_n^{(q)}(t)$ $(n = 1, 2, \ldots)$: these functions will be said to be *associated with* the rectangle.

It will be found that (8) is valid under the conditions just stated provided that, when the point (s, t) is on the boundary of a rectangle such as is described above, each of the various functions occurring in the equations is understood to mean the corresponding function ($\gamma^{(p,q)}(s, t)$, $\psi_n^{(p)}(s)$, etc.) associated with any one rectangle to which (s, t) may be regarded as belonging. Thus the point (a_1, a_1) may be regarded as belonging to four rectangles, and hence (8) is (for $s = a_1$, $t = a_1$) equivalent to

$$\gamma^{(p,q)}(a_1, a_1) = \sum_{n=1} \frac{\psi_n^{(p)}(a_1)\,\psi_n^{(q)}(a_1)}{\mu_n} + \sum_{n=1} \frac{\eta_n^{(p)}(a_1)\,\eta_n^{(q)}(a_1)}{\nu_n}$$

for $p = 1, 2$ and $q = 1, 2$. It will be understood that two or more of the equations just written may be identical, owing to the fact that corresponding functions associated with different rectangles have the same values for $s = a_1$, $t = a_1$.

Each of the series on the right of (8) is absolutely convergent in the square

* Hilbert, *op. cit.*, Chap. 16, may be consulted.

$a \leq s \leq b$, $a \leq t \leq b$, and, further, each is uniformly convergent in the square when $\gamma^{(p,p)}(s, s)$ is continuous in the interval $a_{p-1} \leq s \leq a_p$ for $p = 1, 2, \ldots, N$.

§ 9. Let us now consider Hilbert's function

$$\kappa(s, t) = \alpha(s)\gamma(s, t)\,(a \leq s \leq b,\ a \leq t \leq b),$$

where $\alpha(s)$ is bounded in (a, b) and is continuous at all except a finite number of points of the interval, and $\gamma(s, t)$ is a function of positive type which has the properties specified in § 7.* Since $\kappa(s, t)$ is symmetrisable on the left by $\gamma(s, t)$, it follows from the theorem of § 7 that

$$\gamma(s, t) = \sum_{n=1} \frac{\psi_n(s)\psi_n(t)}{\mu_n} + \sum_{n=1} \frac{\eta_n(s)\eta_n(t)}{\nu_n}.$$

Multiplying through by $\alpha(s)$, it will be found that

$$\kappa(s, t) = \sum_{n=1} \frac{\phi_n(s)\psi_n(t)}{\lambda_n} + \sum_{n=1} \frac{\xi_n(s)\eta_n(t)}{\nu_n}, \tag{9}$$

where $\xi_n(s) = \alpha(s)\eta_n(s)$ and, of course, $\eta_n(s)$ $(n = 1, 2, \ldots)$ constitute an orthogonal system of functions for the interval (a, b), each member of which is orthogonal to $\kappa(s, t)$ on the left.† If we now multiply through the equation (9) by $\eta_m(s)$, and integrate between the limits a and b, we find

$$0 = \sum_{n=1} \frac{\eta_n(t)\int_a^b \xi_n(s)\eta_m(s)\,ds}{\nu_n},$$

whence it appears that $\int_a^b \xi_n(s)\eta_m(s)\,ds = 0$ for all values of n and m. Thus $\kappa(s, t)$ *can be expressed in the form* (9), *where* (i) *the functions* $\eta_n(s)$ $(n = 1, 2, \ldots)$ *constitute an orthogonal system of functions for the interval* (a, b), *and* (ii) *each of the pairs of functions* $\{\xi_n(s), \psi_m(s)\}$, $\{\eta_n(s), \phi_m(s)\}$, $\{\xi_n(s), \eta_m(s)\}$ *is orthogonal for the interval* (a, b) *whatever be the values of* n *and* m.‡

Let it be assumed that $\alpha(s)$ has only one sign in the interval (a, b).§ Since

$$\int_a^b \alpha(s)[\eta_n(s)]^2\,ds = \int_a^b \xi_n(s)\eta_n(s)\,ds = 0$$

it will be clear that $\xi_n(s)$ cannot exist.

* The results obtained are valid when $\gamma(s, t)$ is subject to the wider conditions of § 8.

† The singular values λ_n are not subject to any restriction as to sign: there may be an infinite number of positive singular values and an infinite number of negative singular values.

‡ These are referred to as the orthogonal properties of the functions $\phi_n(s)$, $\psi_n(s)$, $\xi_n(s)$, $\eta_n(s)$, in the next paragraph.

§ $a(s)$ may be zero at points in the interval.

Again, let it be assumed that $\gamma(s, t)$ is definite. It follows from the relation

$$\int_a^b \eta_n(s)\,\kappa(s, t)\,ds = 0,$$

that

$$\int_a^b \xi_n(s)\,\gamma(s, t)\,ds = 0,$$

and hence that $\xi_n(s)$ cannot exist.

It is thus seen that when $\alpha(s)$ has only one sign in $(a\ b)$, or when $\gamma(s, t)$ is definite,

$$\kappa(s, t) = \sum_{n=1} \frac{\phi_n(s)\,\psi_n(t)}{\lambda_n}. \tag{10}$$

It will be clear that the series on the right of (9) are each uniformly convergent in the fundamental square, when $\gamma(s, s)$ is continuous in the interval (a, b).

§ 10. Consider next the function

$$\kappa(s, t) = \int_a^b \alpha(s, x)\,\gamma(x, t)\,dx,$$

where $\alpha(s, t)$ is a symmetric function (not necessarily of positive type) and $\gamma(s, t)$ is a symmetric function of positive type in the square $a \leq s \leq b$, $a \leq t \leq b$. It will be assumed that each of the functions $\alpha(s, t)$ $\gamma(s, t)$ is bounded and summable in the square $a \leq s \leq b$, $a \leq t \leq b$, and that each is continuous with respect to s in the interval (a, b) for any fixed value of t in the interval.* Proceeding as in § 9, it will be found that, because $\kappa(s, t)$ is symmetrisable on the left by $\gamma(s, t)$, *we have the equation* (9), *where now* $\xi_n(s) = \int_a^b \alpha(s, t)\,\eta_n(t)\,dt$, *but the orthogonal properties of the functions* $\phi_n(s)$, $\psi_n(s)$, $\xi_n(s)$, $\eta_n(s)$ *are exactly as before.*

It can be shown that, if $\alpha(s, t)$ is of positive type, or if $\gamma(s, t)$ is definite, the expansion (10) is valid.

The concluding remark at the end of § 9 applies to this case also.

§ 11. The results indicated in the last two paragraphs have important consequences in the theory of the linear integral equation.

For example, when $\kappa(s, t)$ is of either of the two kinds specified, its resolvent is given by

$$\mathrm{K}(\lambda;\, s, t) = \sum_{n=1} \frac{\phi_n(s)\,\psi_n(t)}{\lambda_n - \lambda} + \sum_{n=1} \frac{\xi_n(s)\,\eta_n(t)}{v_n}.$$

In connection with this, it should be observed that the resolvent is symmetrisable on the left by $\gamma(s, t)$.

* Wider conditions, *e.g.*, on the lines of § 8, may be imposed.

Again, the Fredholm determinant is

$$D(\lambda) = \prod_{n=1} \left(1 - \frac{\lambda}{\lambda_n}\right),$$

and is consequently of class ("genre") zero.

Lastly, it may be observed that

$$\sum_{n=1} \frac{1}{\lambda_n} = \int_a^b \kappa(s, s)\, ds.$$

The equation just written leads to interesting results, as to the relation between the singular values of $\kappa(s, t)$ and those of $\gamma(s, t)$.

On the Conditions at the Boundary of a Fluid in Turbulent Motion.

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(Communicated from the National Physical Laboratory. Received April 13, 1920.)

The experiments here described form part of a general research into the phenomena of skin friction of solid surfaces due to the flow over them of fluids whose motion, not in the immediate vicinity of the surface, is eddying or turbulent. Considerable information has been obtained in recent years as to the magnitude of the frictional forces brought into existence in this condition of flow, and the manner of variation of these forces with the relative mean speed of surface and fluid, the roughness of the surface, and the physical characteristics of the fluid is fairly well known. Practically nothing, however, is known about the mechanism by which the resistance to flow is transmitted to the bounding surfaces. For speeds below the critical when the general motion of the fluid throughout is streamline in character, it is generally accepted that the layer of fluid in contact with the boundary is at rest relative to it, as any slipping of finite amount would be detected in a variation from the Poiseuille law of the relationship between the diameter of a pipe and the time of efflux of a given volume of fluid. At speeds above the critical, observations near the walls have shown that the mean velocity falls rapidly as the solid bounding surface is approached, and it has been suggested that at the walls there may exist a thin layer in